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LETTER TO THE EDITOR

The phase boundary in dilute and random Ising and Potts ferromagnets

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Abstract. We present an approach to dilute Ising and Potts models, based on the Fortuin-Kasteleyn random cluster representation, which is simultaneously rigorous, intuitive and surprisingly simple. Our analysis yields, with no dimensional restrictions or other caveats, the following asymptotic form of the phase boundary. For the regular dilute model in which bonds have constant ferromagnetic coupling J with probability p and are vacant with probability $1-p$, the critical temperature scales as $\exp[-J/(kT_c(p))] \sim |p-p_c|$, implying that the crossover exponent is $\Phi = 1$. If the constant couplings are replaced by a distribution $F(J)$ with mass near $J = 0$, quite different crossover behaviour is observed. For example, if $F(J) \sim J^\alpha$ then, for p near p_c, $T_c(p) \sim |p-p_c|^{1/\alpha}$.

The phase diagram of the dilute Ising and Potts models has been the subject of much investigation, both theoretical and experimental, for over a quarter of a century†. In the simplest version of the model, the nearest-neighbour exchange interaction assumes the values J and 0 with probability p and $1-p$, respectively. It is generally expected [1] that as p decreases, the Curie temperature, $T_c(p)$‡, remains non-zero until p reaches the percolation threshold, p_c.

Of particular interest is the behaviour of the model in the vicinity of the point $p = p_c$, $T = 0$. Very early series arguments [3] as well as more sophisticated scaling analyses [4] suggested that this behaviour can be described by a crossover exponent Φ according to

$$\exp\{-J/[kT_c(p)]\} \sim |p-p_c|^\Phi \tag{1}$$

with $\Phi = 1$. While there was no general and completely rigorous derivation, there is a rich history associated with the exponent Φ. The value $\Phi = 1$ was derived by Bergstresser [5] in a paper notable both for its essential rigour and its regrettable lack of impact on later work in the field. Independently, $\Phi = 1$ was predicted to low order [6] and, eventually, to all orders [7] in the ε expansion, where $\varepsilon = 6-d$ and d is the

† For properties of dilute ferromagnets, see [1]. For general properties of the Potts models, both uniform and dilute, see [2].

‡ $T_c(p)$ is usually defined as the threshold where the average (with respect to the random bonds) of the magnetisation at any given site becomes positive. It can be shown that this is also the dividing line between zero and positive magnetisation for almost all bond realisations at every site in the (unique) infinite network of active bonds.

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dimensionality of the lattice. Still later, this crossover behaviour was found in special 'soluble' models [8] and explained by an appealing geometric argument based on the structure of the incipient infinite cluster [9]. In a beautiful experiment [10] the value $\Phi = 1$ was measured in $Rb_2CO_pMg_{1-p}F_4$.

Bergstresser's analysis [5] was based on an elegant differential inequality, the application of which relied on some natural, but unproven assumptions. One of these is that for any p above p_c, there is indeed a Curie transition. Although this was known for some time for p sufficiently close to one [11], the fact that the critical concentration is exactly p_c has only been partially established†. Given the results on the critical concentration, together with Bergstresser's analysis, one obtains a complete derivation of $\Phi = 1$ only in two dimensions [14]. Furthermore, Bergstresser's work covered neither Potts models, nor interactions with a continuum of non-zero values.

In this letter, we present a complete proof of the asymptotics of the phase boundary for the d-dimensional dilute q-state Potts models, which has several advantages over previous treatments. First, the derivation is non-perturbative and thus dispels recent doubts [15] concerning the validity of the result $\Phi = 1$ obtained via the ε expansion (at least for $q \geqslant 1$). Second, and perhaps more importantly, our proof is simpler than previous analyses (both rigorous and non-rigorous), and hence provides some insight into the crossover mechanism. Finally, our analysis also yields the behaviour of systems in which the non-zero interaction J is replaced by a distribution of values; these cases were not treated by previous scaling or geometric arguments, nor considered within the context of the ε expansion. For example, if J is replaced by some distribution $F(J) \sim J^\alpha$ for small J, then the proof yields

$$T_c(p) \sim |p - p_c|^{1/\alpha} \tag{2}$$

in marked contrast to the logarithmic behaviour of $T_c(p)$ seen in (1). The only previous result of this sort seems to be that of Georgii [14] who obtained upper and lower bounds with different powers. It is worth noting that power law (rather than logarithmic) behaviour is characteristic of Heisenberg magnets [1]. Thus the result (2) indicates that the crossover behaviour alone is not sufficient to distinguish between dilute Ising (or Potts) models with 'soft' randomness and dilute Heisenberg models with 'hard' randomness.

The remainder of this letter is devoted to the proof of the statements made above. The proof is based on some intuitive inequalities, derived in [16], for the Fortuin and Kasteleyn [17] random cluster representation of q-state Potts models. We begin with a statement of these bounds, followed by a brief explanation of how they arise naturally within the context of the random cluster representation. We then show how the bounds imply results on dilute and random magnets.

Recall that the q-state Potts model is described by the Hamiltonian

$$\mathscr{H} = -\sum_{\langle i,j\rangle = b \in \mathbb{B}} J_b[\delta(\sigma_i, \sigma_j) - 1] \tag{3}$$

where $\langle i, j\rangle = b$ is a bond between a pair of sites i and j of a regular lattice (or a finite subset thereof), $\mathbb{B}$ is the collection of all such bonds, $\sigma_i = 1, \ldots, q$ is the spin on site i, and $\delta(\sigma_i, \sigma_j) = 1$ if $\sigma_i = \sigma_j$ and zero otherwise. The J_b are bond interactions, which for simplicity‡ we take to be independent and identically distributed non-negative

† For $d = 1$ this was proved by [12], while for $d \geqslant 3$ this was proved, modulo equivalence of a 'slab threshold' with p_c in the pure percolation model, by [13].

‡ One can easily extend the analysis of this letter (with some loss of notational clarity) to include cases other than identically distributed random variables; independence, however, is harder to eliminate.

random variables satisfying $J_b = 0$ with probabilitiy $1-p$, and

$$\text{Prob}[J_b \leqslant J \,|\, J \neq 0] = F(J). \tag{4}$$

We will denote by $M(q, \{J_b\})$ the magnetisation of (the infinite volume limit of) the q-state system with bond interactions $\{J_b\}$.

In [16] bounds were derived which relate the magnetisation in two Potts models with different values of q and $\{J_b\}$. First, for fixed q and any J_b, we define the following two 'density parameters': $\lambda_b \equiv 1-\exp(-\beta J_b)$ and $\lambda_b^* \equiv \lambda_b/[\lambda_b + q(1-\lambda_b)]$, where $\beta = 1/(kT)$ is the inverse temperature. Now consider two models characterised by parameters q, $\{J_b\}$ and q', $\{J_b'\}$ with $q \geqslant q' \geqslant 1$. The domination bounds are: if $\lambda_b' \geqslant \lambda_b$ for every bond of the lattice, then

$$M(q', \{J_b'\}) \geqslant M(q, \{J_b\}) \tag{5a}$$

while if $\lambda_b^{*\prime} \geqslant \lambda_b^*$ for every bond, then

$$M(q', \{J_b'\}) \leqslant M(q, \{J_b\}). \tag{5b}$$

Analogous domination relations are also obeyed by other quantities, e.g. the free boundary condition two-point correlation function.

Although we will not give the complete proofs of these relations here, let us explain how they follow from the random cluster representation. This representation expresses the q-state Potts model as a dependent percolation model with bond density parameter λ_b, as defined above. Indeed, if $\mathbb{G}$ is a subset of $\mathbb{B}$, then the random cluster weight of the graph $\mathbb{G}$ is

$$W(\mathbb{G}) = \prod_{b\in\mathbb{G}} \lambda_b \prod_{b\in\mathbb{B}\backslash\mathbb{G}} (1-\lambda_b) q^{C(\mathbb{G})} \frac{1}{Z(\beta, q)} \tag{6a}$$

where $C(\mathbb{G})$ is the number of connected components of $\mathbb{G}$ and $Z(\beta, q)$ is a normalisation constant (the partition function). In order to count $C(\mathbb{G})$ properly, one must usually specify 'boundary conditions'; see [15] for more details. The bonds of $\mathbb{G}$ will be referred to as 'occupied' while those of $\mathbb{B}\backslash\mathbb{G}$ will be called vacant. Observe that the above weights can be re-expressed† as

$$W(\mathbb{G}) = \prod_{b\in\mathbb{G}} \lambda_b^* \prod_{b\in\mathbb{B}\backslash\mathbb{G}} (1-\lambda_b^*) q^{l(\mathbb{G})} \frac{1}{Z^*(\beta, q)} \tag{6b}$$

where $l(\mathbb{G})$ is the number of independent loops of the graph $\mathbb{G}$ and λ_b^* is defined as above.

Magnetisation in the q-state spin systems can be expressed, probabilistically, in the random cluster picture. Indeed, if the spins along the boundary of $\mathbb{B}$ are set to the spin state $\sigma = 1$, then the magnetisation in the 1-direction of any site i is the probability that i is connected to the boundary by a path of occupied bonds in the associated random cluster model. Hence, in the infinite-volume limit, the spontaneous magnetisation is simply the percolation density of the random cluster system (with 'wired' boundary conditions.)

The domination bounds are simple consequences of expressions (6*a*) and (6*b*), together with the Harris-FKG inequalities [18]; in fact, they are also valid for non-integer q and q', with M interpreted as the percolation density. The first bound follows

† Equation (6*b*) follows easily from (6*a*) by direct calculation. If two endpoints of a bond b are not connected, then the number of connected components is reduced by one if b is occupied. Hence the ratio of these conditional occupation/vacancy probabilitiesis $(\lambda_b/q)/(1-\lambda_0)$. On the other hand, when the endpoints are already connected, occupying the bond forms a loop; hence a factor of q is regained.

from the fact that if the λ_b are held fixed, while q increases, then configurations with more components (hence smaller percolation density) are favoured. Conversely, if the λ_b^* are held fixed, while q increases, then configurations with more loops (hence larger percolation density) are favoured. In particular, these bounds imply that at inverse temperature β, the random cluster representation for a q-state Potts model is less likely to percolate than an independent (i.e. $q=1$) model at density $p=1-\mathrm{e}^{-\beta J}$; on the other hand, it is more likely to do so than the corresponding independent model at bond density $p=(1-\mathrm{e}^{-\beta J})/[1+(q-1)\,\mathrm{e}^{-\beta J}]$. In fact, the preceding sentence constitutes the essence of our analysis of dilute ferromagnets!

Let us restrict attention to $q \geq 1$† in which case we will call the parameter-q random cluster model a 'generalised ferromagnet'. A density-p diluted version of such a system (with or without additional randomness) is, on the one hand, a quenched ferromagnetic system with coupling distribution given by: $J_b=0$ with probability $1-p$, and $\mathrm{Prob}[J_b \leq J \mid J_b \neq 0]=F(J)$; and on the other hand, a random cluster model on a density-p percolation network. For general $q>1$, either point of view seems hopelessly more complicated than a uniform (i.e. non-random) system; however, when $q=1$, the system is trivial. Indeed, it is 'merely' (independent) percolation at bond density

$$\overline{1-\exp\{-\beta J_b\}}=p\int[1-\mathrm{e}^{-\beta J}]\,\mathrm{d}F(J). \tag{7}$$

We may now adopt the second point of view, and apply the domination bounds to relate the systems of interest to $q=1$ models. It follows that the generalised ferromagnetic models on the percolation cluster can be dominated by density $\bar{\lambda}_q \equiv \int[1-\mathrm{e}^{-\beta J}]\,\mathrm{d}F(J)$ *percolation* problem on the percolation network. Similarly, these systems dominate the density $\bar{\lambda}_q^* \equiv \int \mathrm{d}F(J)[1-\mathrm{e}^{-\beta J}]/[1+(q-1)\,\mathrm{e}^{-\beta J}]$ percolation-percolation problems. However, the former *is* percolation at density $\bar{\lambda}_q p$ and the latter *is* percolation at density $\bar{\lambda}_q^* p$. In particular, denoting by $P_\infty(.)$ the percolation density of an ordinary (independent) percolation model, and by $M(q, F, p)$ the magnetisation of the q-state dilute random Potts model, we have‡

$$P_\infty(\bar{\lambda}_q p) \geq M(q, F, p) \geq P_\infty(\bar{\lambda}_q^* p). \tag{8}$$

The rest of this letter is devoted to listing various consequences of equation (8).

(i) For all q-state random and dilute Potts models on regular lattices, with dilution parameter $1-p$, spontaneous magnetisation can occur if and only if p exceeds the percolation threshold, p_c. In particular, this extends the results of [12, 13] to all dimensions without any assumptions concerning the phase structure of ordinary percolation.

(ii) The problem of magnetic systems defined on various 'incipient structures', such as inhomogeneous incipient percolation clusters [19] or the critical cluster of the one-dimensional $1/r^2$ percolation model [20] may be treated by relaxing the requirement that the J_b have identical distribution. In contrast to the result (1) for regular dilute Potts magnets, the analogue of (8) for these systems implies that they are disordered at all non-zero temperatures.

† For $0<q<1$, the Harris-FKG inequalities fail. Even in these cases, somewhat weakened versions of the results presented here may be obtained. However, the usual q tends to zero limit is singular (since one also scales the inverse temperature to zero); without some additional modifications, the domination bounds provide no information.

‡ It should be remarked that the magnetisation appearing here is the *thermodynamic* magnetisation. In general, one obtains the thermodynamic magnetisation only if the finite-volume approximations are computed in systems with the 'right' boundary conditions.

(iii) For the case of simple dilution (i.e. $J=1$ with probability p and $J=0$ otherwise), we have $\bar{\lambda}_q p = p(1-e^{-\beta})$ and $\bar{\lambda}_q^* p = p(1-e^{-\beta})/[1+(q-1)\,e^{-\beta}]$. Evidently $1-e^{-\beta} \leqslant p_c/p$ implies $\beta \leqslant \beta_c(p)$. Hence

$$e^{-\beta_c(p)} \leqslant (p-p_c)/p. \tag{9}$$

Conversely, if $(1-e^{-\beta})/[1+(q-1)\,e^{-\beta}] \geqslant p_c/p$, then $\beta \geqslant \beta_c(p)$. From this, it is not hard to show that

$$\exp[-\beta_c(p)] \geqslant q^{-1}(p-p_c)/p. \tag{10}$$

Notice that (9) and (10) together imply the behaviour (1) with crossover exponent $\Phi = 1$. The rather trivial value of the crossover exponent may now be understood as a consequence of the fact that, for all values of p and β, the relevant quantities in the q-state Potts models are bounded above and below by analogous quantities in independent percolation models.

(iv) For distributions F which are not zero in a neighbourhood of the origin, the asymptotics of the phase boundary are controlled by the behaviour of F near the origin. In particular, assuming that

$$c_1 J^\alpha \leqslant F(J) \leqslant c_2 J^\alpha \tag{11}$$

for J smaller than some J', one has

$$\beta_c(q, F, p) \sim (p-p_c)^{-1/\alpha} \tag{12}$$

where '$\sim$' means there are upper and lower bounds of the stated form with constants depending on q, c_1, c_2 and J'.

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